LONGITUDINAL STREAMLINING OF A SEMIINFINITELY LARGE PLATE BY A VISCOELASTIC FLUID WITH HEAT TRANSFER

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A numerical analysis is made of the dynamic boundary layer and the thermal boundary layer at a semiinfinitely large plate longitudinally streamlined by a viscoelastic fluid.

The first one to solve the equations of a boundary layer for a Newtonian fluid longitudinally streamlining a semiinfinitely large flat plate was Blasius [1]. Subsequently Bairstow [2], Goldstein [3], and Topfer [4] obtained approximate solutions to the Blasius problem. An exact numerical solution was obtained later by Howarth [5]. These results are all included in the well-known monographs by Schlichting [6], Rosenhead [7], and Pai [8]. Owing to advances in technology, there have appeared many new useful fluids. Inasmuch as these fluids exhibit viscoelastic characteristics, they cannot be described on the basis of Navier—Stokes equations. Many researchers have attempted to formulate equations of rheological state for such non-Newtonian fluids. Noted among them should be Oldroyd [9] and Walters [10]. In the latter study fluids with a vanishing memory were considered, known as Walters fluids A' and B'. The equations of a boundary layer in these fluids have been derived by Beard and Walters [11]. On the basis of these equations, they also solved the problem of streamlining of the frontal surface of a blunt body by a viscoelastic fluid. The problem of heat transfer during a flow of this kind was recently solved by Soundalgekar and Vighnasam [12]. As far as the authors know, the problem of flow and heat transfer during longitudinal streamlining of a semiinfinitely large plate by a Walters B' viscoelastic fluid has not yet been solved. Beard and Walters have established, however, that self-adjoint solutions to this problem for a Walters B' fluid can be obtained only covering the frontal stagnation zone. Such solutions do not exist for a longitudinally streamlined semiinfinitely large plate. We therefore propose to solve this system of nonlinear ordinary differential equations numerically.

Mathematical Analysis. Let the x axis lie in the plane of the plate and be oriented in the direction of flow, and let the y axis be normal to it. The equations of rheological state for a Walters B' fluid and the corresponding equations of a boundary layer are [11]: the equations of motion

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-K_0^*\left[u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right], \quad (1)$$

the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (2)$$

the equation of heat (without dissipation)

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}, \quad (3)$$

and the boundary conditions

$$\begin{gathered} u=0,\ v=0,\ T=T_w \ \text{at}\ y=0; \\ u=U_0,\ T=T_\infty \ \text{at}\ y=\infty. \end{gathered} \quad (4)$$

A change to variables

$$\eta=y\sqrt{U_0/\nu x},\ \psi=\sqrt{U_0\nu x}\,f(\eta), \quad (5)$$

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TABLE I. Values of f_0, f_0'', f_1, f_1', f_1''

η	f_0	f_0'	f_0''	f_1	f_1'	f_1''
0,0	0,00000000	0,00000000	0,33205730	0,00000000	0,00000000	—0,19312360
0,2	0,00664099	0,06640778	0,33198380	—0,00378879	—0,03751775	—0,18201008
0,4	0,02655988	0,13276415	0,33146981	—0,01485620	—0,07276827	—0,17035902
0,6	0,05973463	0,19893723	0,38007909	—0,03273428	—0,10558766	—0,15760206
0,8	0,10610821	0,26470911	0,32738923	—0,05691083	—0,13569675	—0,14316088
1,0	0,16557171	0,32977999	0,32300708	—0,08680637	—0,16270333	—0,12649283
1,2	0,23794869	0,39377606	0,31658915	—0,12175272	—0,18611574	—0,10715900
1,4	0,32298154	0,45626171	0,30786536	—0,16097565	—0,20537198	—0,08491051
1,6	0,42032072	0,51675672	0,29666343	—0,20358534	—0,21988713	—0,05978226
1,8	0,52951798	0,57475808	0,28293099	—0,24857800	—0,22911874	—0,03217606
2,0	0,65002430	0,62976566	0,26675152	—0,29485209	—0,23264597	—0,00291009
2,2	0,78119325	0,68131030	0,24835089	—0,34124108	—0,23025319	0,02678845
2,4	0,92229002	0,72898185	0,22809174	—0,38656225	—0,22200504	0,05536426
2,6	1,0725059	0,77245493	0,20645461	—0,42967836	—0,20829755	0,08109046
2,8	1,2309972	0,81150953	0,18400659	—0,46956600	—0,1898716	0,1022844
3,0	1,3968081	0,84604435	0,16136032	—0,50538212	—0,16778025	0,11754818
3,2	1,5690948	0,87608136	0,13912806	—0,53651918	—0,14330896	0,12598570
3,4	1,7469499	0,90176113	0,11787625	—0,56264053	—0,11785941	0,12734769
3,6	1,9295250	0,92332958	0,098086285	—0,58369019	—0,09281329	0,12208046
3,8	2,1160296	0,94111791	0,08012592	—0,59987525	—0,06939925	0,11124824
4,0	2,3057462	0,95551815	0,064234128	—0,61162392	—0,04858450	0,09636581
4,2	2,4980394	0,96695699	0,05051975	—0,61952568	—0,03100730	0,07916666
4,4	2,6923607	0,97587075	0,038972616	—0,62426275	—0,01695781	0,06135975
4,6	2,8882477	0,98268342	0,02948377	—0,62654248	—0,00640469	0,04442064
4,8	3,0853203	0,98778945	0,02187118	—0,62703875	0,00094268	0,02945023
5,0	3,2832733	0,99154182	0,01590680	—0,62634815	0,00555197	0,01711425
5,2	3,4818673	0,99424546	0,01134179	—0,62496332	0,007981214	0,00765896
5,4	3,6809187	0,99615523	0,00792766	—0,62326282	0,00880142	0,00098343
5,6	3,8802903	0,99747769	0,00543195	—0,62151485	0,00853723	—0,00325677
5,8	4,0798815	0,99837542	0,003648414	—0,61989061	0,00762926	—0,00553991
6,0	4,2796205	0,99897280	0,00240204	—0,61848318	0,00641730	—0,00638138
6,2	4,4794569	0,99936246	0,00155017	—0,61732781	0,00514059	—0,00626174
6,4	4,6793562	0,99961162	0,00098061	—0,61642106	0,00394967	—0,00558265
6,6	4,8792954	0,99976779	0,00060804	—0,61573679	0,00292432	—0,00464854
6,8	5,0792593	0,99986374	0,00036956	—0,61523831	0,00209319	—0,00366763
7,0	5,2792383	0,99992153	0,00022016	—0,61488682	0,00145187	—0,00276543
7,2	5,4792263	0,99995564	0,00012857	—0,61464641	0,00097755	—0,00200374
7,4	5,6792196	0,99987538	0,00007359	—0,61448669	0,00063975	—0,00140045
7,6	5,8792159	0,99998658	0,00004129	—0,61438349	0,00040737	—0,00094671
7,8	6,0792139	0,99999280	0,00002270	—0,61431858	0,00025257	—0,00062027
8,0	6,2792129	0,99999620	0,00001224	—0,61427882	0,00015255	—0,00039449
8,2	6,4792123	0,99999801	0,00000646	—0,61425509	0,00008978	—0,00024384
8,4	6,6792120	0,99999896	0,00000334	—0,61424128	0,00005146	—0,00014663
8,6	6,8792119	0,99999945	0,00000170	—0,61423347	0,00002870	—0,00008586
8,8	7,0792118	0,99999969	0,00000846	—0,61422917	0,00001553	—0,00004898

$$u = U_0 f', \quad v = \frac{1}{2}\sqrt{\frac{\nu U_0}{x}}\,(\eta f' - f)$$

transforms Eqs. (1)-(3) to the system of ordinary differential equations

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = \frac{K}{2}\left[\left(\frac{d^2 f}{d\eta^2}\right)^2 - 2\frac{df}{d\eta}\frac{d^3 f}{d\eta^3} - f\frac{d^4 f}{d\eta^4}\right], \tag{6}$$

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{2}\Pr f \frac{d\theta}{d\eta} = 0 \tag{7}$$

with the boundary conditions

$$\begin{aligned} &f = 0,\ f' = 0,\ \theta = 1 \quad \text{at} \quad \eta = 0; \\ &f' = 1,\ \theta = 0 \quad \text{at} \quad \eta \to \infty. \end{aligned} \tag{8}$$

Assuming that $K \ll 1$, we replace the sought functions f and θ with power series in the parameter K. This will result in substantial mathematical simplifications of the fourth-order equation (6) with three boundary conditions. Retaining only the first two terms of the series expansion

$$f = f_0 + K f_1, \quad \theta = \theta_0 + K\theta_1, \tag{9}$$

and inserting them into system (6)-(8), we then equate the corresponding coefficients of the various powers of K (except K^2) and obtain the set of relations

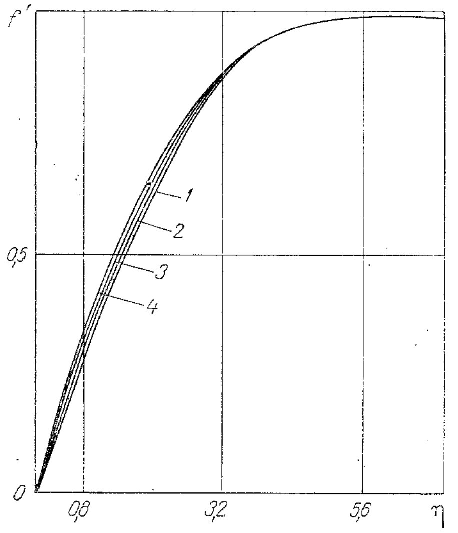

Fig. 1

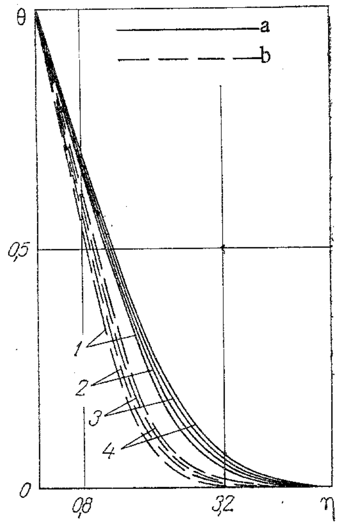

Fig. 2

Fig. 1. Velocity profiles: 1) K = 0; 2) K = 0.05; 3) K = 0.1; 4) K = 0.2.

Fig. 2. Temperature profiles: 1) K = 0; 2) K = 0.05; 3) K = 0.1; 4) K = 0.2; (a) N_{Pr} = 2; (b) N_{Pr} = 5.

$$f_0''' + \frac{1}{2} f_0 f_0'' = 0, \tag{10}$$

$$f_1''' + \frac{1}{2} (f_1 f_0'' + f_0 f_1'') = \frac{1}{2} (f_0''^2 - 2 f_0' f_0''' - f_0 f_0^{IV}), \tag{11}$$

$$\theta_0'' + \frac{1}{2} \Pr f_0 \theta_0' = 0, \tag{12}$$

$$\theta_1'' + \frac{1}{2} \Pr (f_1 \theta_0' + f_0 \theta_1') = 0. \tag{13}$$

The boundary conditions will be

$$\begin{gathered} f_0(0) = 0,\ f_0'(0) = 0,\ \theta_0(0) = 1, \\ f_1(0) = 0,\ f_1'(0) = 0,\ \theta_1(0) = 0, \\ f_0'(\infty) = 1,\ \theta_0(\infty) = 0, \\ f_0'(\infty) = 0,\ \theta_1(\infty) = 0. \end{gathered} \tag{14}$$

Equations (10-(13) have been solved numerically for the boundary conditions (14) and the values obtained for f_0 and f_1 are given in Table 1. The values of $f' = u/U_0$ corresponding to various values of K are given in Table 2. The results are also shown graphically in Fig. 1. It is evident here that the velocity profile becomes broader with higher values of K. The temperature profile in the boundary layer follows the same trend, namely both θ and T increase with K (Fig. 2), but they do not decrease with higher values of the Prandtl number.

It would be interesting to explore the dependence of frictional stresses in the boundary layer on the shear modulus. In the given case

$$\tau_{xy} = \eta_0 \frac{\partial u}{\partial y} - K_0 \left(u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right). \tag{15}$$

At the plate surface (i.e., at y = 0) both u = 0 and v = 0 so that expression (15) reduces to the equality

$$\tau_{xy}\big|_{y=0} = \eta_0 \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{16}$$

TABLE 2. Values of $f' = u/U_0$

η	K=0	K=0,05	K=0,1
0,0	0,00000000	0,00000000	0,00000000
0,2	0,06640778	0,06453189	0,06265601
0,4	0,13276415	0,12912573	0,12548732
0,6	0,19893723	0,19365785	0,18837846
0,8	0,26470911	0,25792427	0,25113943
1,0	0,32977999	0,32164483	0,31350966
1,4	0,45626171	0,44599311	0,43572451
1,8	0,57475808	0,56330214	0,55184620
2,2	0,68131030	0,66979764	0,65828498
2,6	0,77245493	0,76204006	0,75162518
3,0	0,84604435	0,83765534	0,82926633
4,0	0,95551815	0,95308892	0,95065970
5,0	0,99154182	0,99181942	0,99209702
6,0	0,99897280	0,99929366	0,99961453
7,0	0,99992153	0,99999412	1,00006670
8,0	0,99999620	1,00000380	1,00001150
9,0	0,99999981	1,00000020	1,00000060

TABLE 3. Values of $\{-\theta'(0)\}$

K	N_{Pr}	
	2	5
0	0,4223082	0,5766890
0,05	0,4190823	0,5719246
0,1	0,4158565	0,5671602
0,2	0,4094048	0,5576313

or, with the aid of relations (5), to

$$\tau_{xy}\big|_{y=0} = \eta_0 U_0 \sqrt{U_0/\nu x}\, f''(0) = \eta_0 U_0 \sqrt{U_0/\nu x}\,[f_0''(0) + K f_1''(0)]. \tag{17}$$

Values of $f_0''(0)$ and $f_1''(0)$ taken from Table 1 transform expression (17) to the equality

$$\tau_{xy}\big|_{y=0} \doteq \mu U_0 \sqrt{U_0/\nu x}\,(0.3320 - 0.1931\,K). \tag{18}$$

according to which surface friction decreases with increasing shear modulus K.

We will now examine the effect of elasticity on the local thermal fluxes from plate to fluid. From the definition

$$q(x) = -\lambda \left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{19}$$

and relations (5) we obtain

$$q(x) = -\lambda \sqrt{\frac{U_0}{\nu x}}\,(T_w - T_\infty)\,\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0}.$$

The numerical values of $\{-\theta'(0)\}$ corresponding to various values of K and N_{Pr} are given in Table 3. We note that the thermal flux decreases with increasing K, but increases with increasing N_{Pr}.

NOTATION

u and v, longitudinal component and the normal component of velocity; U_0 and T_∞, velocity and the temperature of the oncoming stream; K, elasticity parameter in the Walters B' model; τ_{xy}, shearing stress; q, thermal flux density; μ, dynamic viscosity of the fluid; $\nu = \mu/\rho$, kinematic viscosity of the fluid; $a = \lambda/\rho c_p$, thermal diffusivity of the fluid; $\theta = (T - T_\infty)/(T_w - T_\infty)$, dimensionless temperature drop; and N_{Pr}, Prandtl number; $K = K_0^* U_0/\nu x$.

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ISOTHERMAL FLOW OF A NON-NEWTONIAN FLUID THROUGH THE CHANNEL OF A VOLUTE-TYPE DISK PUMP UNDER CONDITIONS OF COMPLEX SHEAR

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A study is made pertaining to steady laminar flow of an anomalously viscous fluid between two rigid disks in one of which the thread has been cut in the form of an Archimedes spiral.

The advantages of a volute-type disk pump with the thread cut in the form of an Archimedes spiral over a conventional volute-type pump are the simplicity of its construction, the possibility of regulating the clearances between the spiral ridges and the smooth other disk, and the higher pressure head developed. The use of such pumps in industry is not widespread owing to, apparently, not only the large axial forces developing in them (which, by the way, can be successfully reduced by adoption of the bilateral volute construction) but also the unavailability of a design method.

We will consider the isothermal flow of a non-Newtonian fluid through a volute-type disk pump consisting of two parallel rigid disks in one of which the thread has been cut in the form of an Archimedes spiral (Fig. 1a). The threaded disk is stationary, while the smooth disk rotates at a constant angular velocity ω_0. It will be assumed in the formulation of the problem that the channel width S is much larger than the channel depth H and that there are no clearances between the spiral ridges and the smooth disk, the flow of the fluid being steady and laminar. All calculations will refer to the median line of the spiral (dash–dot line on the diagram), considering that the tangential velocity of the smooth disk $V_0 = r\omega_0$ as well as the lead angle of the spiral δ and the pressure gradients $\partial p/\partial \varphi = A_\varphi$, $\partial p/\partial r = A_r$ vary only along the channel (in the φ direction) while remaining constant across its width. Let the inside radius and the outside radius of the Archimedes spiral be r_i and r_o, respectively. The velocity component in the z direction will be disregarded.

In solving this problem we are mostly concerned about the pressure gradients $\partial p/\partial x = A_x$, $\partial p/\partial y = A_y$ and the flow rate Q_x. Accordingly, the vector representing the tangential velocity of the smooth disk V_0 can be resolved into two components: $V_x = V_0 \cos\delta$ and $V_y = V_0 \sin\delta$ (Fig. 1b).

The equations of motion, in projection on the axes φ and r, can be written as

$$\frac{\partial \tau_{\varphi z}}{\partial z} = \frac{A_\varphi}{r}, \quad \frac{\partial \tau_{rz}}{\partial z} = A_r - \rho\,\frac{V_\varphi^2}{r}. \tag{1}$$

An analysis of the solution to Eqs. (1) for a Newtonian fluid has revealed that, with

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